

Exercises on dynamic programming and optimal control
 Doctoral Advanced Macroeconomics, Fall 2025
 Instructed by Ming Yi
 Due on Jan/25/2026

1. We are trying to solve the following growth model:

$$\text{Lifetime utility : } U = \sum_{t=0}^{\infty} (\beta)^t \log c_t, \quad (1)$$

$$\text{subject to } k_{t+1} = Ak_t^\alpha - c_t. \quad (2)$$

- (a) i. You are asked to do the “guess-and-verify” exercise. First, let us guess the **value function** and **policy function** of the Bellman Equation for the dynamic programming problem above as:

$$\text{value function : } V(k_t) = \lambda + \xi \log k_t, \quad (3)$$

$$\text{policy function : } k_{t+1} = \pi(k_t) = \gamma Ak_t^\alpha, \quad (4)$$

Then, you should use the Euler equations in your lecture notes to prove the following statements

$$\xi = \frac{\alpha}{1 - \alpha\beta}, \quad \lambda = \frac{\log[A(1 - \alpha\beta)]}{1 - \beta} + \frac{\alpha\beta \log(A\alpha\beta)}{(1 - \alpha\beta)(1 - \beta)}, \quad \gamma = \alpha\beta.$$

- ii. Given $\beta = 0.99$, $\alpha = 0.2$, $A = 2$. Based on your results in (a), draw the value function and policy function out, using *your favorite software*. What is the steady-state capital stock k^* and consumption c^* ?
- (b) Restate the problem above in the form of Problem A2 and Problem A3 as in the lecture notes.
- (c) Do the following steps (**notice: make sure that you know why we are doing the following steps! If not, you should double-check the lecture notes and recall that the Bellman function actually constructs a contraction mapping!**):
- Define the maximum and minimum values k can take as a 90% deviation from the steady state value of k (we are not interested in all feasible value of k). Next, create a vector of length $N = 1000$ as the grid values k can take, bounded by the minimum and maximum values you have just calculated. Let us denote that vector k with elements $k(1) < k(2) < \dots < k(N)$, with $k(1)$ equal to the minimum value and $k(N)$ equal to the maximum value.

- Pick a small value ϵ as the convergence criterion (any number you think sensible). A number too small will take you forever to run the program and a number too big will give you inaccurate estimates. Let the initial guess for the value function to be $V_0(k) \equiv 0$ for any k .
- For each $i = 1, \dots, N$, find the $k(j)$ that maximizes $\log[2k(i)^{0.2} - k(j)]$ (recall that $V_0 \equiv 0$). Make sure that you do not pick a $k(j)$ that makes consumption negative, for all $i = 1, \dots, N$. Then keep the maximum value as $V_1(i)$ and memorize the “position” j (i.e., which grid value of capital you have picked above while solving the maximization problem).

After you have done the maximization problem above for all $i = 1, \dots, N$, you should have a $N \times 1$ vector of maximum values V_1 and a $N \times 1$ vector of policy π_1 , which contains the “position” of the grid value of capital you have picked. The policy π_1 tells you what next period capital you should pick given the current capital: $k_{t+1} = \pi_1(k_t)$.

- For each $i = 1, \dots, N$, find the $k(j)$ that maximizes $\log[2k(i)^{0.2} - k(j)] + \beta V_1(j)$. Then keep the maximum value as $V_2(i)$ and memorize the “position” j (i.e., which grid value of capital you have picked above while solving the maximization problem). After you have done the maximization problem above for all $i = 1, \dots, N$, you should have a $N \times 1$ vector of maximum values V_2 and a $N \times 1$ vector of policy π_2 , which contains the “position” of the grid value of capital you have picked. The policy π_2 tells you what next period capital you should pick given the current capital: $k_{t+1} = \pi_2(k_t)$.
- Repeat the above step many times, until

$$\max_j \{|V_n(j) - V_{n-1}(j)|\} < \epsilon. \quad (5)$$

That is, the iterative algorithm continues until the largest absolute difference between the corresponding elements for the two value functions is less than ϵ .

Your program has converged to the fixed point!

Now, treat V_n as your value function and π_n as your policy function obtained from the iterative method above. Plot the two functions with the grid values of k on the x -axis. Are the functions the same as those you found in (a)?

- Repeat (b) with a lower discount rate $\beta = 0.8$. How does the slope of the policy function change? What does that mean *in words*?
- Repeat (b) with the CRRA utility function $\mu(c_t) = \frac{c_t^{0.5} - 1}{0.5}$ and $\beta = 0.99$. What difference does the new functional form make?
- Repeat (b) with a bigger $\epsilon = 0.001$. Can you find any differences?

2. Consider the problem below:

$$\begin{aligned} & \max_{\{k(t), c(t)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \log c(t) \\ & \text{subject to } k(t+1) = k(t)^\alpha - c(t), \\ & k(0) > 0, \beta \in (0, 1) \end{aligned}$$

- (a) Method 1: Guess the policy function as $\pi(x) = \gamma x^\alpha$, and verify your guess by determining the value of γ . (economic intuition?)
- (b) Method 2: Guess the value function as $V(x) = \lambda + \xi \log x$, and verify your guess by determining the values of λ and ξ .
- You should find that the two methods above are equivalent.

3. Consider the following problem:

$$\max_{[c(t), a(t)]_{t=0}^1} \int_0^1 e^{-\rho t} u(c(t)) dt, \quad (6)$$

$$\text{subject to } \dot{a}(t) = ra(t) + \omega - c(t), \quad a(0) = a_0, \quad a(1) = 0. \quad (7)$$

where r and ω are exogenously defined constants.

- (a) Deduce the Euler-Lagrange equation for the problem above.
- (b) Rearrange your result above to give the Euler equation usually used in your textbooks, $\frac{u''(c(t))\dot{c}(t)}{u'(c(t))} = \rho - r$, namely, along the household's optimal path, the growth rate of its marginal utility of consumption should be equal to the gap between the discount rate ρ and interest rate r .
- (c) Use the Pontryagin's Maximum Principle (Theorem 4 in your lecture notes) to get the same results.
- (d) Given $u(c) = \log(c)$, can you **solve** the problem above? What if $u(c) = [\theta - e^{-\beta c(t)}]$?