Advanced Macroeconomics

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Midterm Exam I (Open-Book)

Undergraduate Program in Economics, HUST

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Name: ______________________ **Student ID:**______________________

1. $(10' + 10' + 10' + 20' + 10' + 10' = 70$ points) In your textbook, knowledge enters the production function in a *labor-augmenting* way, i.e., $Y(t) = F(K(t), A(t)L(t))$. Recall that knowledge can also enter the production in a *capital-augmenting* way: $Y(t) = F(A(t)K(t), L(t))$. For simplicity, let us further assume the production function takes the Cobb-Douglas form:

$$F(AK, L) = (AK)^{\alpha} L^{1-\alpha}, \tag{1}$$

where $\alpha \in (0, 1)$. All other settings in your textbook remain here: Capital depreciates with rate δ, labor grows with rate n, the saving rate is s, and knowledge progresses with rate g.

(a) Does the production function depicted in (1) satisfy the Constant Returns to Scale property and Inada conditions?

(b) It is said that, because the knowledge in this model is "capital-augmenting", it can only improve the productivity of capital, and does not change labor's productivity. True or False? Explain your answers.

(c) What is the appropriate intensive-form production function, $y = f(k)$, for this model?

(d) Prove that this model admits a unique steady state, and find the expression for the steady-state k^*.

(e) What is the growth rate of output per capita along the *balanced growth path*?

(f) Now, change the production function to a new one with *Hicks-neutral* knowledge:

$$Y = A \cdot F(K, L) = AK^{\alpha}L^{1-\alpha}, \tag{2}$$

with all other settings the same as above. Redo questions (c)–(e).

2. ($10' + 20' = 30$ points) Answer the following questions regarding materials in Part A of Chapter 2.

(a) The model assumes a constant discount rate for instantaneous utility, i.e., $\rho(t) \equiv \rho$. As a result, $e^{-\rho t}u(C(t)$ tells as how to discount time-t utility levels to time-0 parallels. However, if we allow $\rho(t)$ to vary with time, e.g., to be an integrable function (可积函数) over any finite time interval, how would you discount $u(C(t))$ to time-0?

(b) Using equations (2.22)–(2.23) in your textbook, the author tries to derive the *Euler equation* intuitively: Along the optimal path, a representative household has no incentive to transfer a small amount of consumption, Δc, from date t to $t + \Delta t$. Replace Δc by ΔC, and derive the Euler equation again.

1.(a) $F(cAK, cL) = (cAK)^{\alpha}(cL)^{1-\alpha}$
$= c(AK)^{\alpha}L^{1-\alpha}$
$= cF(AK, L)$

$\frac{\partial F}{\partial K} = \alpha A^{\alpha}K^{\alpha-1}L^{1-\alpha} > 0$

$\lim\limits_{K\to 0}\frac{\partial F}{\partial K} = \infty$, $\lim\limits_{K\to\infty}\frac{\partial F}{\partial K} = 0$

$\frac{\partial F}{\partial L} = (1-\alpha)(AK)^{\alpha}L^{-\alpha} > 0$

$\lim\limits_{L\to 0}\frac{\partial F}{\partial L} = \infty$ $\lim\limits_{L\to\infty}\frac{\partial F}{\partial L} = 0$

又 $\frac{\partial^2 F}{\partial K^2} = \alpha(\alpha-1)A^{\alpha}K^{\alpha-2}L^{1-\alpha} < 0$

$\frac{\partial^2 F}{\partial L^2} = -\alpha(1-\alpha)(AK)^{\alpha}L^{-\alpha-1} < 0$

∴ 两个条件都满足.

(b) 错误 $\frac{\partial F}{\partial L} = (1-\alpha)(AK)^{\alpha}L^{-\alpha}$

所以A的增加也会增加劳动的边际生产力.

(c) 令 $A_L = A^{\frac{\alpha}{1-\alpha}}$ 则 $g_L = \frac{\alpha}{1-\alpha}g$ 令 $\mu = \frac{\alpha}{1-\alpha}$: $g_L = \mu g$

∴ $y = \frac{F}{A_L L} = \frac{(AK)^{\alpha}L^{1-\alpha}}{A^{\frac{\alpha}{1-\alpha}}L} = \left(\frac{K}{A_L L}\right)^{\alpha} = k^{\alpha}$

(d) $\dot{k}(t) = \frac{\partial\left[\frac{K(t)}{A_L(t)L(t)}\right]}{\partial t}$

$= \frac{\dot{K}(t)}{A_L(t)L(t)} - \frac{K(t)}{A_L(t)L(t)}\left(\frac{\dot{A}_L(t)}{A_L(t)} + \frac{\dot{L}(t)}{L(t)}\right)$

$= \frac{sF(AK,L) - \delta K}{A_L(t)L(t)} - \frac{K(t)}{A_L(t)L(t)}\left(\frac{\dot{A}_L(t)}{A_L(t)} + \frac{\dot{L}(t)}{L(t)}\right)$

$= sf(k) - (\delta + n + \mu g)k$.

∵ $\cancel{f''(k) < 0,\ \lim\limits_{k\to 0} f(k) = \infty\ \lim\limits_{k}}$

∵ $f'(k) > 0$ $f''(k) < 0$

所以存在 k^*. 使得 $\dot{k}(t) = 0$. 即收敛于稳态

由 $sf(k^*) = (\delta + n + \mu g)k^*$

$k^* = \left(\frac{s}{\delta + n + \mu g}\right)^{\frac{1}{1-\alpha}}$

(e) k^* 不变. 则 $f(k)$ 不变.

∴ $\frac{F}{L} = A_L f(k)$ 以 $\frac{\alpha}{1-\alpha}g$ 的速度增长.

(f) 令 $A_L = A^{\frac{1}{1-\alpha}}$ $g_L = \frac{1}{1-\alpha}g$. $k = \frac{K}{A_L L}$.

则 $y = \frac{F}{A_L L} = \frac{AK^{\alpha}L^{1-\alpha}}{A^{\frac{1}{1-\alpha}}L} = \left(\frac{K}{A^{\frac{1}{1-\alpha}}}\right)^{\alpha} = k^{\alpha}$

$k^* = \left(\frac{s}{\delta + n + \frac{1}{1-\alpha}g}\right)^{\frac{1}{1-\alpha}}$

$\frac{F}{L}$ 的增长率为 $\frac{1}{1-\alpha}g$

$$e^{-\int_0^t \rho(z)dz}u(C(t))$$

2\. (a) $\cancel{e^{-\int_0^t \rho(t)dt}u(C(t))}$

(b) $$U=\frac{L(0)}{H}\int_0^\infty e^{-(\rho-n)t}\frac{C(t)^{1-\theta}}{1-\theta}dt$$

$$u'(c)=\cancel{\frac{L(0)}{H}}\ \frac{L(0)}{H}C(t)^{-\theta}e^{-(\rho-n)t}$$

在t时期减少 ΔC 的消费，~~成~~效用损失为

$$\frac{L(0)}{H}e^{-(\rho-n)t}C(t)^{-\theta}\Delta C$$

在 $t+\Delta t$ 时期，能够增加 $e^{(r(t)-n)\Delta t}\Delta C$ 的消费

dc的边际效用为

$$\frac{L(0)}{H}e^{-(\rho-n)(t+\Delta t)}\left(C(t)e^{\frac{\dot{C}(t)}{C(t)}\Delta t}\right)^{-\theta}$$

由效用等价可得

$$\frac{L(0)}{H}e^{-(\rho-n)t}C(t)^{-\theta}\Delta C=\frac{L(0)}{H}e^{-(\rho-n)(t+\Delta t)}\left(C(t)^{-\theta}\left(e^{\frac{\dot{C}(t)}{C(t)}\Delta t}\right)^{-\theta}e^{(r(t)-n}\right.$$

化简得 $$1=e^{-(\rho-n)\Delta t}e^{-\theta\frac{\dot{C}(t)}{C(t)}\Delta t}e^{(r(t)-n)\Delta t}$$

同时取对数后除以 Δt 可得

$$\frac{\dot{C}(t)}{C(t)}=\frac{r(t)-\rho}{\theta}$$