Applied Game Theory
Graduate Program in Economics, HUST
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HOMEWORK #1

1. Suppose $n \geq 2$ students are attending a course at HUST, in the current semester. For student i, her effort put into the course is $e_i \in (0, 100)$. We assume $e_1 > e_2 > \cdots > e_n$. The grade of each student is determined exclusively by the final exam of the course.

   - Each student has the opportunity to choose if she cheats in the exam, i.e., the action set for student i is $S_i = \{H, C\}$, where action H stands for behaving honestly in the exam (not cheating), and action C represents cheating in the exam.
   - If student i does not cheat in the exam, i.e., $s_i = H$, the exam leads to a *fair* result: Student i receives e_i as her grade.
   - If student i cheats in the exam, she will be caught by the proctor with probability $\lambda \in [0, 1]$ and a punishment of $P > 0$ points incurs; otherwise she successfully promote her grade by $Q > 0$ points. In other words, a cheating student i receives grade $e_i + Q$ with probability $(1 - \lambda)$, and gets grade $e_i - P$ with probability λ. For simplicity, the grades are allowed to be negative or greater than 100.
   - Given the realizations of all grades, these numbers will be arranged in a descending order (from the highest to the lowest). If two students have the same grade, the student with greater effort is assigned with a higher ranking. Students only care about their rankings of grades. What is more, the students are risk-neutral individuals: Given $s \in S$, if player figures out that she will be the xth player with probability γ and be the yth player with probability $(1 - \gamma)$, she will think of herself being arranged at the position with ranking $\gamma x + (1 - \gamma) y$, i.e., her *expected* ranking.
   - It is thus straightforward to see that every student has incentive to *minimize* its expected ranking, which is allowed to be any number in $\mathbb{R}_+$, rather than the set of all natural numbers, $\mathbb{N}$.

   (a) Given $\lambda = 0$. Under what conditions C is a weakly dominant strategy for each student?

(b) Given $\lambda = 0$. Could $s^* = (s_j^*)$ with $s_j^* = H$ for all $j = 1, \cdots, n$ be a Nash equilibrium for any specific game described above? If yes, give an example of the specific game. If no, prove it.

(c) Given $\lambda \in (0, 1]$. Can you find a threshold $\bar{P}$, such that it is a strictly dominant strategy for all students to play H, as long as $P > \bar{P}$?

2. Consider a model of noncooperative behavior in cooperative production with $k \geq 2$ individuals. The production function is

$$\pi(s_1, \cdots, s_k) = \sqrt{\sum_{i=1}^{k} s_i}\ ,$$

the payoff function for player i is $\mu_i(s_i, x_i) = x_i - s_i$, where x_i denotes player i's share of the production π. Specifically, the *proportional rule* is adopted here, i.e., player i gets:

$$x_i(s) = \begin{cases} 0, & \text{if } \ s = 0; \\ \frac{s_i}{\sum_{j=1}^{k} s_j}\pi(s), & \text{if } \ s > 0. \end{cases} \tag{1}$$

(a) Show that at a pure Nash Equilibrium $s^* = (s_1^*, \cdots, s_k^*)$,

(i) $\sum_j s_j^* > 0$;

(ii) $s_1^* = \cdots = s_k^*$.

(b) Determine all pure Nash equilibria.

(c) You should obtain a single pure Nash equilibrium of the form $s^* = (r, \cdots, r)$ with $r = r(k)$.

(iii) Show that $r \cdot k$ is increasing in k with limit 1.

(iv) Show that r is decreasing in k with limit 0.

3. Consider a Cournot game with two or more firms. In the following of the problem, let $a > 0$ and let the inverse demand function be given by

$$P(Q) = \begin{cases} a - Q, & \text{for } \ 0 \leq Q \leq a; \\ 0, & \text{for } \ Q > a. \end{cases}$$

Let $n \geq 2$ be the number of firms. Firm i has marginal cost $c_i \geq 0$. It chooses output $q_i \geq 0$ and incurs costs $c_i \cdot q_i$. Total or industry output is $Q = \sum_{j=1}^{n} q_j$. Firm i's payoff (profit) is $\pi_i(q_1, \cdots, q_n) = P(Q) \cdot q_i - c_i \cdot q_i$.

(a) Suppose $n = 2$ and $c_1 \leq c_2$. Show that $q_1^* \geq q_2^*$ at each Nash equilibrium (q_1^*, q_2^*).

(b) Suppose $n > 2$ and $1 < c_1 \leq c_2 \leq \cdots \leq c_n$. Consider a Nash equilibrium $(q_1^*, \cdots, q_n^*)$. Show that

  (i) $Q^* = \sum_j q_j^* < a$.

  (ii) $q_1^* \geq q_2^* \geq \cdots \geq q_n^*$.

(c) For arbitrary n, show that there exists a unique Nash equilibrium $(q_1^*, \cdots, q_n^*)$ with $q_i^* > 0$ for all i.

4. Consider a 2-person zero-sum game with payoff matrix given in Table 1.

Table 1: The payoff matrix of a zero-sum game.

		Player 2		
		ℓ	m	r
Player 1	U	$1, -1$	$1, -1$	$-1, 1$
	M	$1, -1$	$1, -1$	$1, -1$
	D	$-1, 1$	$1, -1$	$1, -1$

(a) Give the definition of a maximin or prudent strategy of a player.

(b) Determine the unique maximin strategy (in pure and mixed strategies) of player 1. Explain it.

(c) Using your answer in (b), find all Nash equilibria (in pure and mixed strategies).